

An Effective Potential for Composite Operators

YUE HU

*Department of Physics
Columbia University
New York, NY 10027*

ABSTRACT

We study the effective potential for composite operators. Introducing a source coupled to the composite operator, we define the effective potential by a Legendre transformation. We find that in three or fewer dimensions, one can use the conventionally defined renormalized operator to couple to the source. However, in four dimensions, the effective potential for the conventional renormalized composite operator is divergent. We overcome this difficulty by adding additional counterterms to the operator and adjusting these order by order in perturbation theory. These counterterms are found to be non-polynomial. We find that, because of the extra counterterms, the composite effective potential is gauge dependent. We display this gauge-dependence explicitly at two-loop order.

1. Introduction

It is well known that the effective potential for elementary fields is gauge dependent[1]. The effective potential can be used in the studies of spontaneous symmetry break, inflationary cosmology and many other problems. It is important to examine if the gauge dependence of the effective potential causes the physical quantities to be gauge dependent. Nielsen discovered an important identity on the gauge-dependence of the effective potential[2]. With the Nielsen identity and its variations, many physical quantities can be proved to be gauge independent.

The gauge dependence of the effective potential arises because the elementary fields are not invariant under gauge transformation. This suggests that one might obtain an explicitly gauge-independent result by defining an effective potential for a gauge-invariant composite operator[3]. We will examine this issue in this paper. To find the effective potential $U(\sigma)$ for a composite operator φ^2 [4], one introduces a source coupled to this operator

$$\mathcal{L} \rightarrow \mathcal{L} - J\varphi^2. \quad (1.1)$$

The effective potential $U(\sigma)$ is the Legendre transform of the ground state energy density $w(J)$ with constant external source J , where σ is conjugate to J . Since the unrenormalized composite operator φ^2 is divergent in general, one has to renormalize it by adding appropriate counterterms to the unrenormalized operator. In this paper, we will use the following method to calculate the effective potential for the renormalized operator $[\varphi^2]$: We will couple the system to two external sources

$$\mathcal{L} \rightarrow \mathcal{L} - J[\varphi^2] - h\varphi = \mathcal{L}(J) - h\varphi. \quad (1.2)$$

If we treat the first external source as part of the Lagrangian, and Legendre transform with respect to the other source h , we will get the effective potential for the modified Lagrangian, $V_{\mathcal{L}(J)}(\phi)$. We introduce an intermediate object

$$Y(\sigma, J, \phi) = J\sigma + V_{\mathcal{L}(J)}(\phi). \quad (1.3)$$

We will show that by minimizing $Y(\sigma, J, \phi)$ with respect to J and ϕ , one can get the effective potential $U(\sigma)$ from the function $Y(\sigma, J, \phi)$. In three and fewer dimensions, if we use the

renormalized operator $[\varphi^2]$, the effective potential $U(\sigma)$ is a well-defined finite object. However, in four dimensions, there does not exist a finite effective potential for the conventionally defined composite operator $[\varphi^2]$. This is because in four dimensions, matrix elements with more than one insertion of $[\varphi^2]$ is divergent in general even when we have added appropriate counterterms to the composite operator to make matrix elements with one insertion finite. Therefore, adding a source of $[\varphi^2]$ causes vacuum energy divergence in the composite effective potential. We define a finite composite effective potential by adding extra counterterms to $[\varphi^2]$ and adjusting these counterterms order by order. We find that at two-loop order, the counterterms of the modified composite operator become non-polynomial.

Although the ordinary effective potential is gauge dependent, its minimal value is gauge independent because of the Nielsen identity[2]. Therefore, the composite effective potential is gauge independent because it is the minimal value of $Y(\sigma, J, \phi)$ with respect to J and ϕ . In four dimensions, the extra counterterms of the modified composite operator invalidate this argument. If we adopt a minimal subtraction scheme for the counterterms, we find that the composite effective potential is explicitly gauge dependent at two-loop order. However, one may add finite counterterms to the composite operator to make the composite effective potential gauge independent, but there is no preferred description to choose finite parts over the other descriptions. Hence although we can make the composite effective potential gauge independent, there is no clear description in how to resolve the arbitrariness.

In Section II, we study the physical meanings of the effective potential and the calculation method that we will use later. In Section III, we illustrate our method with a three-dimensional example. In Section IV, we show the need to add extra counterterms in four dimensions and that the extra counterterms are non-polynomial by studying an ungauged $O(N)$ model. In Section V, we use the same method to study the effective potential $U(\sigma)$ for scalar QED and its gauge dependence.

2. Effective Potential and Calculation Method

Let us consider a quantum field theory in Euclidean space at zero temperature. For simplicity, we will restrict ourselves to the case where there is only one elementary field. The generalization to multiple fields is straightforward. Introducing an external source $J(x)$ to the elementary field $\varphi(x)$, we can define a functional of $J(x)$

$$W[J] = \ln \int [\mathcal{D}\varphi] e^{-S + \int d^d x J(x)\varphi(x)} \quad (2.1)$$

where S is the Euclidean action of the theory and d is the number of space-time dimensions. We define a new variable $\phi(x)$ by

$$\phi(x) = \frac{\delta W[J]}{\delta J(x)}. \quad (2.2)$$

The effective action of the theory is defined as the Legendre transformation of the functional $W[J]$

$$\Gamma[\phi] = \int d^d x J(x)\phi(x) - W[J]. \quad (2.3)$$

We can expand the effective action as a power series of external momenta. In position space, such an expansion can be written as

$$\Gamma[\phi] = \int d^d x \left[V(\phi) + \frac{1}{2}(\partial\phi)^2 \Gamma^{(1)}(\phi) + \dots \right]. \quad (2.4)$$

The function $V(\phi)$ in the first term of this expansion is called the effective potential for the elementary field φ .

Suppose H is the Hamiltonian of the system. Let us ask the following question: among all states that satisfy the constraint

$$\langle \psi | \varphi(x) | \psi \rangle = \phi, \quad (2.5)$$

which state has the minimal value of $\langle \psi | H | \psi \rangle$ and what is the minimal value? Using the method of Lagrange multiplier, one finds that the minimal value is related to the effective

potential by

$$V(\phi) = J\phi - \frac{W}{\beta\Omega} = \frac{\langle H \rangle_{\min}}{\Omega}. \quad (2.6)$$

Thus we conclude that, at zero temperature, if we require that a homogeneous system satisfy the condition $\langle \varphi(x) \rangle = \phi$, the minimal energy density of the system is the effective potential $V(\phi)$.

To calculate the effective potential $V(\phi)$, let us shift the field and define a new field $\tilde{\varphi}$ by

$$\varphi(x) = \tilde{\varphi}(x) + \phi. \quad (2.7)$$

Next we rewrite the Lagrangian in terms of the new field $\tilde{\varphi}(x)$ and separate out the terms linear in $\tilde{\varphi}$:

$$\mathcal{L} = \mathcal{L}' + a_1 \tilde{\varphi}(x). \quad (2.8)$$

By the definition of the effective potential, we have

$$e^{-\int d^d x V(\phi)} = e^{W - \int d^d x J\phi} = \int [\mathcal{D}\varphi] e^{-\int \mathcal{L} + \int J\varphi - \int J\phi}. \quad (2.9)$$

In terms of the new variable $\tilde{\varphi}(x)$, this equation can be written as

$$e^{-\int V(\phi)} = \int [\mathcal{D}\tilde{\varphi}] e^{-\int \mathcal{L}' + \int (J - a_1)\tilde{\varphi}} = e^{W_{\mathcal{L}'}[J - a_1]}. \quad (2.10)$$

The subscript \mathcal{L}' of W means that we use \mathcal{L}' here as the Lagrangian to calculate the generating functional W . Since the state $|\psi\rangle$ satisfies the constraint Eq. (2.5), we have

$$\tilde{\phi}_{\mathcal{L}'}(x) \Big|_{J=a_1} = \frac{\delta W_{\mathcal{L}'}[J]}{\delta J(x)} \Big|_{J=a_1} = \langle \psi | \tilde{\varphi}(x) | \psi \rangle = 0. \quad (2.11)$$

Hence we have

$$-\int V(\phi) = \left[\int (J - a_1) \tilde{\phi}_{\mathcal{L}'} - \int V_{\mathcal{L}'}(\tilde{\phi}_{\mathcal{L}'}) \right] \Big|_{\tilde{\phi}_{\mathcal{L}'}=0} = -\int V_{\mathcal{L}'}(0) \quad (2.12)$$

which in turn leads to

$$V(\phi) = V_{\mathcal{L}'}(0). \quad (2.13)$$

(This is just another form of Jackiw's original result[6].) Thus, the vacuum energy in the

theory with Lagrangian \mathcal{L}' is equal to the effective potential $V(\phi)$. Therefore, we only need to sum over one-particle irreducible graphs with no external fields. For any given number of loops, there are only a limited number of such graphs.

The effective potential can be similarly defined for a composite operator $O(x)$. (We only consider local operators $O(x)$. For treatments of non-local operators, see [5].) Adding an external source $\int d^d x J(x) O(x)$ to the system, we can define a functional of $J(x)$

$$W[J] = \ln \int [\mathcal{D}\varphi] e^{-S + \int d^d x J(x) O(x)} \quad (2.14)$$

and its Legendre transform

$$\begin{aligned} \sigma(x) &= \frac{\delta W[J]}{\delta J(x)} \\ \Gamma[\sigma] &= \int d^d x J(x) \sigma(x) - W[J]. \end{aligned} \quad (2.15)$$

When $\sigma(x)$ is a constant, the functional $\Gamma[\sigma]$ can be written as

$$\Gamma[\sigma] = \int d^d x U(\sigma). \quad (2.16)$$

The function $U(\sigma)$ is called the effective potential for the composite operator $O(x)$. As for the case of the ordinary effective potential, we can show that the effective potential $U(\sigma)$ is the minimal energy density of the system under the constraint

$$\langle O(x) \rangle = \sigma. \quad (2.17)$$

We can introduce external sources for both $O(x)$ and $\varphi(x)$ and define the effective potential $V(\sigma, \phi)$ by a double Legendre transformation. Writing

$$\mathcal{L}(J) = \mathcal{L} - JO, \quad (2.18)$$

we define

$$\begin{aligned} \widetilde{W}[J, h] &= \ln \int [\mathcal{D}\varphi] e^{-\int \mathcal{L} + \int (JO + h\varphi)} \\ &= \ln \int [\mathcal{D}\varphi] e^{-\int \mathcal{L}(J) + \int h\varphi}. \end{aligned} \quad (2.19)$$

and its Legendre transform

$$\tilde{\Gamma}[\sigma, \phi] = \int (J\sigma + h\phi) - \widetilde{W}[J, h] \quad (2.20)$$

where the new variables $\sigma(x)$ and $\phi(x)$ are defined by

$$\begin{aligned} \sigma(x) &= \frac{\delta \widetilde{W}[J, h]}{\delta J(x)}, \\ \phi(x) &= \frac{\delta \widetilde{W}[J, h]}{\delta h(x)}. \end{aligned} \quad (2.21)$$

In this paper we will transform with respect to h first to get

$$\Gamma'[J, \phi] = \int h\phi - \widetilde{W}[J, h] = \Gamma_{\mathcal{L}(J)}[\phi] \quad (2.22)$$

Then we transform with respect to J , obtaining

$$\begin{aligned} \tilde{\Gamma}[\sigma, \phi] &= \int J\sigma + \Gamma'[J, \phi], \\ \sigma(x) &= - \frac{\delta \Gamma'[J, \phi]}{\delta J(x)}. \end{aligned} \quad (2.23)$$

It is easy to see that these two approaches are equivalent to each other.

IF $\sigma(x)$ and $\phi(x)$ are constant, $J(x)$ and $h(x)$ must be also, and we can write

$$\begin{aligned} \tilde{\Gamma}[\sigma, \phi] &= \int d^d x V(\sigma, \phi), \\ \Gamma'[J, \phi] &= \int d^d x V_{\mathcal{L}(J)}(\phi). \end{aligned} \quad (2.24)$$

where function $V_{\mathcal{L}(J)}(\phi)$ is the ordinary effective potential with $\mathcal{L}(J)$ as the Lagrangian. From Eq. (2.23), we find

$$V(\sigma, \phi) = J\sigma + V_{\mathcal{L}(J)}(\phi) \quad (2.25)$$

If we treat J as independent of σ and ϕ , we can consider the right side of the above equation

as a new function $Y(\sigma, J, \phi)$ of three variables

$$Y(\sigma, J, \phi) = J\sigma + V_{\mathcal{L}(J)}(\phi). \quad (2.26)$$

By the properties of Legendre transformation, $V(\sigma, \phi)$ only depends on two independent variables σ and ϕ . Thus to get $V(\sigma, \phi)$ from $Y(\sigma, J, \phi)$, we must have

$$\frac{\partial Y(\sigma, J, \phi)}{\partial J} = 0. \quad (2.27)$$

If we set the external source $h(x)$ for $\varphi(x)$ to be zero, the function $V(\sigma, \phi)$ reduces to the effective potential $U(\sigma)$ for the operator O . This condition is equivalent to

$$\frac{\partial V(\sigma, \phi)}{\partial \phi} = 0. \quad (2.28)$$

The function $V(\sigma, \phi)$ is the minimal energy density among all states that satisfy the constraints $\langle O(x) \rangle = \sigma$ and $\langle \varphi(x) \rangle = \phi$, while the function $U(\sigma)$ is the minimal energy density among states that satisfy only the single constraint $\langle O(x) \rangle = \sigma$. Thus $U(\sigma)$ is the minimum of $V(\sigma, \phi)$ for all values of ϕ . Expressed in terms of the function $Y(\sigma, J, \phi)$, the above condition becomes

$$\frac{\partial Y(\sigma, J, \phi)}{\partial \phi} = 0. \quad (2.29)$$

After we have solved Eqs. (2.27) and (2.29) for $J(\sigma)$ and $\phi(\sigma)$ from, the effective potential $U(\sigma)$ for the composite operator is just

$$U(\sigma) = Y(\sigma, J(\sigma), \phi(\sigma)). \quad (2.30)$$

In summary, we can define a function $Y(\sigma, J, \phi)$ by Eq. (2.26). Using minimization conditions Eqs. (2.27) and (2.29), we can determine the functions $J(\sigma)$ and $\phi(\sigma)$. Substituting these into $Y(\sigma, J, \phi)$, we can find out the effective potential $U(\sigma)$ as in Eq. (2.30).

3. Three-dimensional Examples

In this section, we will use some examples in three dimensions to illustrate our method. Let us consider a theory with Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial\varphi_1)^2 + \frac{1}{2}(\partial\varphi_2)^2 + \frac{1}{2}m^2(\varphi_1^2 + \varphi_2^2) + \frac{\lambda}{4}(\varphi_1^2 + \varphi_2^2)^2 + \frac{\kappa}{6}(\varphi_1^2 + \varphi_2^2)^3 + \text{counterterms.} \quad (3.1)$$

In three dimensions, λ has mass dimension one and κ is dimensionless. In the $\overline{\text{MS}}$ scheme, which we use, only the mass counterterm, which is linear in m^2 , depends on the mass parameter. Thus the renormalized φ^2 operator is

$$[\varphi^2] = 2\frac{\partial\mathcal{L}}{\partial m^2} = \varphi_1^2 + \varphi_2^2 + \text{counterterms.} \quad (3.2)$$

We will use an external source $\frac{1}{2}J[\varphi^2]$ that is proportional to the mass term. Thus, $\mathcal{L}(J)$ in Eq. (2.18) is the original Lagrangian with m^2 replaced by $m^2 - J$ and $V_{\mathcal{L}(J)}(\phi)$ is the ordinary effective potential with m^2 replaced by $m^2 - J$. (In the following, we will use $\frac{1}{2}\sigma$ instead of σ in the formalism so that σ corresponds to the expectation value of $[\varphi^2]$.)

First we will find the ordinary effective potential $V(\phi)$. Following the method that we discussed earlier, we shift the fields by a constant amount

$$\begin{aligned} \varphi_1(x) &= \tilde{\varphi}_1(x) + \phi, \\ \varphi_2(x) &= \tilde{\varphi}_2(x). \end{aligned} \quad (3.3)$$

The shifted Lagrangian without the terms linear in $\tilde{\varphi}$ is

$$\begin{aligned} \mathcal{L}' &= \frac{1}{2}(\partial\tilde{\varphi}_1)^2 + \frac{1}{2}(\partial\tilde{\varphi}_2)^2 + \frac{1}{2}(m^2 + 3\lambda\phi^2 + 5\kappa\phi^4)\tilde{\varphi}_1^2 + \frac{1}{2}(m^2 + \lambda\phi^2 + \kappa\phi^4)\tilde{\varphi}_2^2 \\ &\quad + \frac{1}{2}m^2\phi^2 + \frac{1}{4}\lambda\phi^4 + \frac{1}{6}\kappa\phi^6 + \text{interaction terms} + \text{counterterms.} \end{aligned} \quad (3.4)$$

To one-loop order, the effective potential is finite and equal to

$$\begin{aligned} V(\phi) &= \frac{1}{2}m^2\phi^2 + \frac{1}{4}\lambda\phi^4 + \frac{1}{6}\kappa\phi^6 \\ &\quad - \frac{1}{12\pi} \left[(m^2 + 3\lambda\phi^2 + 5\kappa\phi^4)^{3/2} + (m^2 + \lambda\phi^2 + \kappa\phi^4)^{3/2} \right]. \end{aligned} \quad (3.5)$$

Replacing m^2 by $m^2 - J$ in $V(\phi)$ gives $V_{\mathcal{L}(J)}(\phi)$. The function $Y(\sigma, J, \phi)$ is related to this

by Eq. (2.26). To one-loop order, we have

$$Y(\sigma, J, \phi) = \frac{1}{2}J\sigma + \frac{1}{2}(m^2 - J)\phi^2 + \frac{\lambda}{4}\phi^4 + \frac{\kappa}{6}\phi^6 - \frac{1}{12\pi} \left[(m^2 - J + 3\lambda\phi^2 + 5\kappa\phi^4)^{3/2} + (m^2 - J + \lambda\phi^2 + \kappa\phi^4)^{3/2} \right]. \quad (3.6)$$

Applying Eqs. (2.27) and (2.29) to this function, we get

$$\begin{aligned} \phi(m^2 - J + \lambda\phi^2 + \kappa\phi^4) - \frac{\phi}{4\pi} \left[(3\lambda + 10\kappa\phi^2)(m^2 - J + 3\lambda\phi^2 + 5\kappa\phi^4)^{1/2} \right. \\ \left. + (\lambda + 2\kappa\phi^2)(m^2 - J + \lambda\phi^2 + \kappa\phi^4)^{1/2} \right] = 0 \end{aligned} \quad (3.7)$$

and

$$\sigma - \phi^2 + \frac{1}{4\pi} \left[(m^2 - J + 3\lambda\phi^2 + 5\kappa\phi^4)^{1/2} + (m^2 - J + \lambda\phi^2 + \kappa\phi^4)^{1/2} \right] = 0. \quad (3.8)$$

One solution to these equations is

$$\begin{aligned} \phi &= 0 \\ (m^2 - J)^{1/2} &= -2\pi\sigma. \end{aligned} \quad (3.9)$$

Substituting this into $Y(\sigma, J, \phi)$ gives

$$U(\sigma) = Y(\sigma, J(\sigma), \phi(\sigma)) = \frac{1}{2}m^2\sigma - \frac{2}{3}\pi^2\sigma^3. \quad (3.10)$$

As we can see from Eq. (3.9), this is valid for $\sigma < 0$. (σ can be negative because we have subtracted a divergent number from it to make it finite.) Since $\phi = 0$, the state corresponding to this solution is in the symmetric phase with $\langle \varphi(x) \rangle = 0$. In the above equation, the first term is the classical value, and the second term comes from zero-point energy of quantum oscillators around the origin.

A second, non-trivial, solution with $\phi \neq 0$ can be obtained by solving the equations order by order. At tree-level, Eqs. (3.7) and (3.8) give

$$\begin{aligned}\phi^2 &= \sigma, \\ J &= m^2 + \lambda\sigma + \kappa\sigma^2.\end{aligned}\tag{3.11}$$

Substituting these relations back into Eqs. (3.7) and (3.8), and keeping terms to one-loop order, we get

$$\begin{aligned}\phi^2 &= \sigma + \frac{1}{4\pi}(2\lambda\sigma + 4\kappa\sigma^2)^{1/2}, \\ J &= m^2 + \lambda\sigma + \kappa\sigma^2 - \frac{1}{2\pi}(\lambda + 4\kappa\sigma)(2\lambda\sigma + 4\kappa\sigma^2)^{1/2}.\end{aligned}\tag{3.12}$$

The effective potential in this case is

$$U(\sigma) = \frac{1}{2}m^2\sigma + \frac{1}{4}\lambda\sigma^2 + \frac{1}{6}\kappa\sigma^3 - \frac{1}{12\pi}(2\lambda\sigma + 4\kappa\sigma^2)^{3/2}.\tag{3.13}$$

This is valid for $\sigma > 0$. Since $\phi \neq 0$, the state corresponding to this solution is in an asymmetric phase.

As σ approaches zero from below and from above, Eq. (3.10) and (3.13) give

$$\lim_{\sigma \rightarrow 0_-} U(\sigma) = \lim_{\sigma \rightarrow 0_+} U(\sigma) = 0 \quad \lim_{\sigma \rightarrow 0_-} \frac{dU(\sigma)}{d\sigma} = \lim_{\sigma \rightarrow 0_+} \frac{dU(\sigma)}{d\sigma} = \frac{m^2}{2}.\tag{3.14}$$

Therefore the effective potential and its first derivative are continuous at where the symmetric and asymmetric solutions connect.

We have plotted the effective potential $U(\sigma)$ versus σ and the ordinary effective potential $V(\phi)$ versus ϕ^2 for the case where $m^2 < 0$, $\lambda > 0$ and $\kappa = 0$ in Figure 1. The reason that they look similar is that both are dominated by the tree-level contributions, which are the same for both cases. There are some important differences. Their one-loop order corrections are different. More importantly, $U(\sigma)$ is real everywhere, while $V(\sigma)$ has an imaginary part for small ϕ^2 if $m^2 < 0$. While $V(\phi^2)$ is only defined for $\phi^2 > 0$, the effective potential $U(\sigma)$ is defined for all values of σ . Moreover, the ordinary effective potential $V(\phi)$ has a potential barrier for small ϕ , while the effective potential $U(\sigma)$ is a globally convex function without any potential barrier in the case of φ^4 theory.

When $m^2 > 0$, $\kappa > 0$ and λ is an appropriately chosen negative quantity, the ordinary effective potential $V(\sigma)$ has a local minimum at $\phi = 0$ and other local minima away from the origin. In this case, the effective potential $U(\sigma)$ has two minima, one corresponding to $\phi = 0$, and the other to $\phi \neq 0$, as shown in Figure 2. The effective potential $U(\sigma)$ now has a potential barrier between these two minima and it becomes complex in this region. It is compared to the ordinary effective potential in Figure 3.

Each point on $U(\sigma)$ and $V(\phi)$ represents a state. A point on $U(\sigma)$ represents a state that has the minimal energy density among all states that satisfy the constraint $\langle [\varphi^2] \rangle = \sigma$, and a point on $V(\phi)$ represents a state that has the minimal energy density among all states that satisfy the constraint $\langle \varphi \rangle = \phi$. One may ask, for appropriately selected values of σ and ϕ , whether $U(\sigma)$ and $V(\phi)$ represent the same state. We shall now examine if there is a correspondence between them. For $V(\phi)$, only the point $\phi = 0$ corresponds to a state in the symmetric phase. For $U(\sigma)$, all points that satisfy $\sigma \leq \sigma_0$ for some value of σ_0 are in the symmetric phase. Therefore, for $\sigma \leq \sigma_0$, except for the one point of $U(\sigma)$ that corresponds to the point $\phi = 0$ of $V(\phi)$, no points of $U(\sigma)$ map to $V(\phi)$. Now let us consider the asymmetric phase. For any value of σ , $U(\sigma)$ is the minimal energy density among all states that satisfy the constraint $\langle [\varphi^2] \rangle = \sigma$. Similarly, for any value of ϕ , $V(\phi)$ is the minimal energy density among all states that satisfy the constraint $\langle \varphi \rangle = \phi$. If the state represented by $\langle [\varphi^2] \rangle = \sigma$ with minimal energy density $U(\sigma)$ has expectation value $\langle \varphi \rangle = \phi$, we must have $V(\phi) \leq U(\sigma)$, since $V(\phi)$ is the minimal energy density among all states that satisfy the constraint $\langle \varphi \rangle = \phi$. For the state represented by $\langle \varphi \rangle = \phi$ with minimal energy $V(\phi)$, its expectation value $\langle [\varphi^2] \rangle = \sigma'$ is different from σ in general, unless the state represented by $\langle \varphi \rangle = \phi$ with energy density $V(\phi)$ is the same as the state represented by $\langle [\varphi^2] \rangle = \sigma$ with energy density $U(\sigma)$. If these two states are the same, there is a mapping between $U(\sigma)$ and $V(\phi)$, and we have $\sigma = \sigma'$ and $U(\sigma) = V(\phi)$. If they are not the same, then we must have $U(\sigma') < V(\phi)$ since $U(\sigma')$ is the minimal energy density among all states which satisfy the constraint $\langle [\varphi^2] \rangle = \sigma'$. Thus we conclude that there is a mapping between $U(\sigma)$ and $V(\phi)$ if and only if $\sigma' = \sigma$.

First, let us find $\langle \varphi \rangle = \phi$ for the state represented by $\langle [\varphi^2] \rangle = \sigma$ with minimal energy density $U(\sigma)$. The argument ϕ in the function $Y(\sigma, J, \phi)$ is the expectation value that we

are looking for. To one-loop order, the value of ϕ is given by Eq. (3.12).

We want to find the expectation value $\langle[\varphi^2]\rangle = \sigma'$ for the state represented by $\langle\varphi\rangle = \phi$ with minimal energy density $V(\phi)$. As we showed earlier, the renormalized composite operator $[\varphi^2]$ is related to the Lagrangian by

$$[\varphi^2] = 2\frac{\partial\mathcal{L}}{\partial m^2}. \quad (3.15)$$

The state in question is a vacuum state for Lagrangian

$$\mathcal{L}' = \mathcal{L} - J\varphi \quad (3.16)$$

where J is a parameter to be determined and does not depend on the space-time variable x . Hence

$$\langle[\varphi^2]\rangle = \frac{\int[\mathcal{D}\varphi]2\frac{\partial\mathcal{L}}{\partial m^2}e^{-\int\mathcal{L}+\int J\varphi}}{\int[\mathcal{D}\varphi]e^{-\int\mathcal{L}+\int J\varphi}}. \quad (3.17)$$

Integrating this gives

$$\begin{aligned} \int d^n x \langle[\varphi^2]\rangle &= \frac{-2\frac{\partial}{\partial m^2} \int[\mathcal{D}\varphi]e^{-\int\mathcal{L}+\int J\varphi}}{\int[\mathcal{D}\varphi]e^{-\int\mathcal{L}+\int J\varphi}} = -2\frac{\partial W[J]}{\partial m^2} \\ &= 2\frac{\partial\Gamma[\phi]}{\partial m^2} = 2 \int d^n x \frac{\partial V(\phi)}{\partial m^2}. \end{aligned} \quad (3.18)$$

We conclude that

$$\sigma' = \langle[\varphi^2]\rangle = 2\frac{\partial V(\phi)}{\partial m^2}. \quad (3.19)$$

To one-loop order, the ordinary effective potential is given by Eq. (3.5). Using Eq. (3.19) and neglecting terms of two-loop order or higher, we find that

$$\sigma' = \phi^2 - \frac{1}{4\pi} \left[(m^2 + 3\lambda\phi^2 + 5\kappa\phi^4)^{1/2} + (m^2 + \lambda\phi^2 + \kappa\phi^4)^{1/2} \right]. \quad (3.20)$$

Using Eqs. (3.12) and (3.20), we find that, to one-loop order, σ and σ' are related by

$$\begin{aligned}\sigma' = \sigma &+ \frac{1}{4\pi}(2\lambda\sigma + 4\kappa\sigma^2)^{1/2} \\ &- \frac{1}{4\pi} \left[(m^2 + 3\lambda\sigma + 5\kappa\sigma^2)^{1/2} + (m^2 + \lambda\sigma + \kappa\sigma^2)^{1/2} \right].\end{aligned}\tag{3.21}$$

At the local minima and local maxima of $U(\sigma)$, the equation

$$m^2 + \lambda\sigma + \kappa\sigma^2 = 0\tag{3.22}$$

holds at tree-level and $\sigma' = \sigma$. We see that to one-loop order σ' differs from σ unless $m^2 + \lambda\sigma + \kappa\sigma^2 = 0$ at tree-level. Therefore, we find that except at local minima and local maxima, there is no mapping between $U(\sigma)$ and $V(\phi)$ in the asymmetric phase.

4. Ungauged $O(N)$ Model in Four Dimensions

In three or fewer dimensions, the vacuum energy counterterm is either linear in m^2 or independent of m^2 . In four dimensions, however, it is quadratic in m^2 . Adding a source term $\frac{1}{2}J[\varphi^2]$ causes a divergence in the effective potential because the vacuum energy divergence is not cancelled. In this section, we will use an ungauged $O(N)$ model to study this problem and find a solution.

The Lagrangian of our $O(N)$ model is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi_a\partial_\mu\varphi_a + \frac{1}{2}m^2\varphi_a\varphi_a + \frac{\lambda}{8N}(\varphi_a\varphi_a)^2 + \text{counterterms}.\tag{4.1}$$

Although we will not use the large- N limit, we can use powers of N to organize our results.

Of all the counterterms, only the mass and vacuum energy counterterms depend on m^2 , and they depend on it in a simple way (so-called soft-parameterization[8]). To all orders, we can write the mass terms as

$$\frac{1}{2}m^2\varphi_a\varphi_a \left(1 + \sum_{i=1}^{\infty} b_i\lambda^i \right)\tag{4.2}$$

where b_i 's are simple poles in $4 - d$. Similarly, the vacuum energy counterterms can be

written as

$$\frac{m^4}{4} \sum_{i=1}^{\infty} c_i \lambda^{i-1} \quad (4.3)$$

where c_i 's are also simple poles in $4 - d$. All other terms are independent of m^2 in our $\overline{\text{MS}}$ scheme.

By differentiating the Lagrangian with respect to the renormalized mass parameter m^2 , we can get the renormalized composite operator $[\varphi_a \varphi_a]_{\text{r}}$:

$$[\varphi_a \varphi_a]_{\text{r}} = 2 \frac{\partial \mathcal{L}}{m^2} = \varphi_a \varphi_a \left(1 + \sum b_i \lambda^i \right) + m^2 \sum c_i \lambda^{i-1}. \quad (4.4)$$

This operator is finite in the sense that all matrix elements with one insertion of this operator are finite. However, matrix elements with more than one insertion of this operator are in general divergent. This can be seen from the fact that the generating functionals we get by adding external sources coupled to this operator are divergent.

Consider the Lagrangian obtained by replacing m^2 in Eq. (4.1) with $m^2 - J$. Only the mass and vacuum energy terms are affected by this replacement. The mass term becomes

$$\frac{1}{2} (m^2 - J) \varphi_a \varphi_a \left(1 + \sum b_i \lambda^i \right), \quad (4.5)$$

and the vacuum term becomes

$$\frac{(m^2 - J)^2}{4} \sum c_i \lambda^{i-1}. \quad (4.6)$$

The Lagrangian differs from the original one by a term linear in J

$$-\frac{1}{2} J \left[\varphi_a \varphi_a \left(1 + \sum b_i \lambda^i \right) + m^2 \sum c_i \lambda^{i-1} \right] = -\frac{1}{2} J [\varphi_a \varphi_a]_{\text{r}} \quad (4.7)$$

and a term quadratic in J

$$\frac{J^2}{4} \sum c_i \lambda^{i-1}. \quad (4.8)$$

Therefore, if we added these two terms as sources to the original Lagrangian, we would get a finite theory with mass parameter $m^2 - J$. However, we are not allowed to add source terms

quadratic in J . If we only add the source term of Eq. (4.7), the generating function $W[J]$ will be divergent because of the lack of the divergent J^2 terms in Eq. (4.8). Consequently, the effective potential for the operator $[\varphi_a \varphi_a]_r$ will be divergent.

In order for the generating functional $W[J]$ to be finite, the matrix elements with any number of insertions of the source term must be finite. It is not possible to find such a composite operator for all states. However, the effective potential is the minimal energy density under the given constraint. We only need to find a composite operator such that for the state with the minimal energy density, the matrix elements with any number of insertions of this operator are finite. If we add extra terms to the composite operator $[\varphi_a \varphi_a]_r$, it is possible to cancel the J^2 divergence at the minimizing point only. So we define a new composite operator

$$[\varphi_a \varphi_a]_s = [\varphi_a \varphi_a]_r + \sum f_i(\varphi_a \varphi_a). \quad (4.9)$$

With this new operator, the function $Y(\sigma, J, \phi)$ is still divergent. The effective potential $U(\sigma)$ obtained by applying the minimization conditions Eqs. (2.27) and (2.29) to $Y(\sigma, J, \phi)$ will be finite. We will adjust the coefficient functions f_i 's order by order in perturbation theory so that the divergences in the effective potential for the composite operator $[\varphi_a \varphi_a]_s$ are cancelled.

Adding a source

$$-\frac{1}{2}J[\varphi_a \varphi_a]_s \quad (4.10)$$

to the original $O(N)$ Lagrangian $\mathcal{L}(\varphi; m^2)$ with mass parameter m^2 , gives a new Lagrangian

$$\mathcal{L}(J) = \mathcal{L}(\varphi; m^2 - J) - \frac{1}{2}J \sum \left[f_i(\varphi_a \varphi_a) + \frac{1}{2}J c_i \lambda^{i-1} \right]. \quad (4.11)$$

The ordinary effective potential then becomes

$$V_{\mathcal{L}(J)}(\phi) = V(\phi; m^2 - J) + V_{\text{ex}}(J, \phi) \quad (4.12)$$

where $V(\phi; m^2 - J)$ is the ordinary effective potential but with $m^2 - J$ as its mass parameter and $V_{\text{ex}}(J, \phi)$ arises from the last term in Eq. (4.11), either directly or through insertion into

a larger graph. From $V_{\mathcal{L}(J)}(\phi)$, we obtain

$$Y(\sigma, J, \phi) = \frac{1}{2}J\sigma + V_{\mathcal{L}(J)}(\phi). \quad (4.13)$$

We will adjust the f_i 's so that after minimization with respect to J and ϕ , the function $Y(\sigma, J, \phi)$ yields a finite effective potential $U(\sigma)$.

We must first obtain the ordinary effective potential and some of the counterterms. Shifting the fields by $\varphi_a(x) = \tilde{\varphi}_a(x) + \phi\delta_{aN}$, we find that, up to one-loop order in the $\overline{\text{MS}}$ scheme, the effective potential is

$$\begin{aligned} V(\phi) = & \frac{1}{2}m^2\phi^2 + \frac{\lambda}{8N}\phi^4 + \frac{(m^2 + \frac{3\lambda}{2N}\phi^2)^2}{64\pi^2} \left(\ln \frac{m^2 + \frac{3\lambda}{2N}\phi^2}{\bar{\mu}^2} - \frac{3}{2} \right) \\ & + (N-1) \frac{(m^2 + \frac{\lambda}{2N}\phi^2)^2}{64\pi^2} \left(\ln \frac{m^2 + \frac{\lambda}{2N}\phi^2}{\bar{\mu}^2} - \frac{3}{2} \right). \end{aligned} \quad (4.14)$$

The one-loop order vacuum energy counterterm is

$$c_1 = \frac{N}{16\pi^2\epsilon}. \quad (4.15)$$

The two-loop order contributions can be similarly calculated. The two-loop order vacuum energy counterterm is

$$c_2 = \frac{(N+2)}{2(4\pi)^4\epsilon^2}. \quad (4.16)$$

Now we are ready to study the effective potential for $[\varphi_a\varphi_a]_s$. To one-loop order, we have

$$\begin{aligned} Y(\sigma, J, \phi^2) = & \frac{1}{2}J\sigma + \frac{1}{2}(m^2 - J)\phi^2 + \frac{\lambda}{8N}\phi^4 + \frac{(m^2 - J + \frac{3\lambda}{2N}\phi^2)^2}{64\pi^2} \left(\ln \frac{m^2 - J + \frac{3\lambda}{2N}\phi^2}{\bar{\mu}^2} - \frac{3}{2} \right) \\ & + (N-1) \frac{(m^2 - J + \frac{\lambda}{2N}\phi^2)^2}{64\pi^2} \left(\ln \frac{m^2 - J + \frac{\lambda}{2N}\phi^2}{\bar{\mu}^2} - \frac{3}{2} \right) - \frac{1}{2}J \left[f_1(\phi^2) + \frac{1}{2}Jc_1 \right]. \end{aligned} \quad (4.17)$$

(For convenience, we will use ϕ^2 instead of ϕ as one of Y 's argument.) At tree-level, the minimization conditions Eqs. (2.27) and (2.29) give $J_0 = m^2 + \frac{\lambda}{2N}\sigma$ and $\sigma(\phi^2)_0 = \sigma$. Sub-

stituting this back into $Y(\sigma, J, \phi^2)$, we find that to one-loop order the effective potential is

$$\begin{aligned}
U(\sigma) &= Y_0(\sigma, J_0, (\phi^2)_0) + \left. \frac{\partial Y_0}{\partial J} \right|_{J_0, \phi_0^2} J_1 + \left. \frac{\partial Y_0}{\partial (\phi^2)} \right|_{J_0, \phi_0^2} (\phi^2)_1 + Y_1(\sigma, J_0, (\phi^2)_0) \\
&= \frac{1}{2} m^2 \sigma + \frac{\lambda}{8N} \sigma^2 + \frac{1}{64\pi^2} \left(\frac{\lambda \sigma}{N} \right)^2 \left(\ln \frac{\lambda \sigma}{N \bar{\mu}^2} - \frac{3}{2} \right) - \frac{1}{2} J_0 \left[f_1(\sigma) + \frac{1}{2} J_0 c_1 \right] \\
&= \frac{1}{2} m^2 \sigma + \frac{\lambda}{8N} \sigma^2 + \frac{1}{64\pi^2} \left(\frac{\lambda \sigma}{N} \right)^2 \left(\ln \frac{\lambda \sigma}{N \bar{\mu}^2} - \frac{3}{2} \right).
\end{aligned} \tag{4.18}$$

The second and third terms on the first line vanish because of the minimization conditions Eqs. (2.27) and (2.29). To make $U(\sigma)$ finite to this order, we must have

$$f_1(\sigma) = -\frac{1}{2} J_0 c_1 = -\frac{N}{32\pi^2 \epsilon} \left(m^2 + \frac{\lambda}{2N} \sigma \right). \tag{4.19}$$

(There is no finite term in above equation because of the $\overline{\text{MS}}$ scheme we use.) Using this f_1 , we apply the minimization conditions Eqs. (2.27) and (2.29) and find that to one-loop order

$$\begin{aligned}
J &= J_0 + J_1 = m^2 + \frac{\lambda \sigma}{2N} + \frac{\lambda^2 \sigma}{16\pi^2 N^2} \left(\ln \frac{\lambda \sigma}{N \bar{\mu}^2} - 1 \right), \\
(\phi^2) &= (\phi^2)_0 + (\phi^2)_1 = \sigma - \frac{\lambda \sigma}{16\pi^2 N} \left(\ln \frac{\lambda \sigma}{N \bar{\mu}^2} - 1 \right) - \frac{N}{32\pi^2 \epsilon} \left(m^2 + \frac{\lambda}{2N} \sigma \right).
\end{aligned} \tag{4.20}$$

Notice that U and J are finite functions of σ , while (ϕ^2) is a divergent function of σ .

Expanding Y to two-loop order, we find that the two-loop order contribution to the $U(\sigma)$ is

$$\begin{aligned}
U_2(\sigma) &= \frac{1}{2} J_1 (\phi^2)_1 - \frac{\lambda}{8N} (\phi^2)_1^2 + V_2((\phi^2)_0; m^2 - J_0) \\
&\quad - \frac{1}{2} \left(m^2 + \frac{\lambda}{2N} \sigma \right) \left[f_2(\sigma) + \frac{1}{2} \lambda c_2 \left(m^2 + \frac{\lambda}{2N} \sigma \right) \right] + U'_{\text{ex}}(\sigma).
\end{aligned} \tag{4.21}$$

where c_2 is the two-loop order vacuum energy counterterm coefficient given in Eq. (4.16)

and the $U'_{\text{ex}}(\sigma)$ term comes from insertions of $-\frac{1}{2}Jf_1(\varphi_a\varphi_a)$ in one-loop graphs:

$$U'_{\text{ex}}(\sigma) = \frac{\lambda^2\sigma}{8N(4\pi)^4} \left(m^2 + \frac{\lambda}{2N}\sigma \right) \left[-\frac{1}{\epsilon^2} + \frac{1}{\epsilon} \left(\ln \frac{\lambda\sigma}{N\bar{\mu}^2} - 1 \right) \right] + \text{finite terms.} \quad (4.22)$$

Hence

$$U_2(\sigma) = -\frac{1}{2} \left(m^2 + \frac{\lambda}{2N}\sigma \right) \left\{ f_2(\sigma) + \frac{\lambda^2\sigma}{2N(4\pi)^4\epsilon} \left(\ln \frac{\lambda\sigma}{N\bar{\mu}^2} - 1 \right) + \frac{\lambda}{4(4\pi)^4\epsilon^2} \left[\frac{\lambda\sigma}{N} + \left(\frac{5}{4}N + 2 \right) \left(m^2 + \frac{\lambda\sigma}{2N} \right) \right] \right\} + \text{finite terms} \quad (4.23)$$

To make this finite, we must have

$$f_2(\sigma) = -\frac{\lambda}{4(4\pi)^4\epsilon^2} \left[\frac{\lambda\sigma}{N} + \left(\frac{5}{4}N + 2 \right) \left(m^2 + \frac{\lambda\sigma}{2N} \right) \right] - \frac{\lambda^2\sigma}{2N(4\pi)^4\epsilon} \left(\ln \frac{\lambda\sigma}{N\bar{\mu}^2} - 1 \right) \quad (4.24)$$

in our minimal subtraction scheme. Notice that the function $f_2(\varphi_a\varphi_a)$ not only has terms proportional to m^2 and $\varphi_a\varphi_a$, but also has terms logarithmic in $\varphi_a\varphi_a$. Thus, beginning at two-loop order, the counterterms in $[\varphi_a\varphi_a]_s$ become non-polynomial.

Our method is only applicable for the asymmetric solution. In that case, both J and ϕ vary with σ , so adding appropriate counterterms can cancel the divergence proportional to J^2 . For the symmetric solution, ϕ is a constant and so we cannot cancel this divergence by adding counterterms. We have been unable to find a way to define a finite composite effective potential in the symmetric phase.

5. Scalar QED in Four Dimensions

In this section, we will use the method demonstrated above to study the effective potential for $[\varphi_a \varphi_a]_s$ for scalar QED in four dimensions. As in the case of ungauged $O(N)$ model in four dimensions, we will need to add extra counterterms to the conventionally defined operator $[\varphi_a \varphi_a]_r$ to obtain a finite composite effective potential. We will also examine the gauge dependence of the composite effective potential.

The Lagrangian of scalar QED is

$$\mathcal{L} = \frac{1}{2} D_\mu \varphi_a D_\mu \varphi_a + \frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \frac{1}{2} m^2 \varphi^2 + \frac{\lambda}{4} \varphi^4 \quad (5.1)$$

where $\varphi^2 = \varphi_1^2 + \varphi_2^2$, $\varphi^4 = (\varphi^2)^2$ and $\varphi_a (a = 1, 2)$ are real fields. We will use the R_ξ -gauge with a gauge fixing term

$$\frac{1}{2\xi} (\partial \cdot A + e v \cdot \varphi)^2 \quad (5.2)$$

where v_a is an external 2-vector. This gauge fixing term requires a ghost compensating term

$$\partial_\mu c^* \partial_\mu c + e^2 (v \times \varphi) c^* c. \quad (5.3)$$

The theory has two dimensionful parameters, m^2 and v . Of the counterterms, only the mass and the vacuum energy counterterms depend on m^2 . We can write these as

$$\frac{1}{2} m^2 \varphi_a \varphi_a \left(1 + \sum_{i=1}^{\infty} b_i(\lambda, e^2) \right) + m^2\text{-independent terms}, \quad (5.4)$$

and

$$\frac{m^4}{4} \sum_{i=1}^{\infty} c_i(\lambda, e^2) + \frac{1}{2} m^2 v^2 \sum_{i=1}^{\infty} d_i(\lambda, e^2) + m^2\text{-independent terms} \quad (5.5)$$

where the b_i 's are polynomials of λ and e^2 of order i , and the c_i 's and d_i 's are polynomials of λ and e^2 of order $i - 1$. The coefficients of these polynomials are simple poles in $4 - d$.

As in the case of the ungauged $O(N)$ model in four dimension, the effective potential for the conventionally defined composite operator $[\varphi_a \varphi_a]_r$ is divergent. We need to use a composite operator with additional counterterms,

$$[\varphi_a \varphi_a]_s = [\varphi_a \varphi_a]_r + \sum f_i(\varphi_a \varphi_a). \quad (5.6)$$

With this coupled to a source, we have

$$\mathcal{L}(J) = \mathcal{L}(\varphi, A, c^*, c; m^2 - J) - \frac{1}{2}J \sum \left[f_i(\varphi_a \varphi_a) + \frac{1}{2}J c_i \right]. \quad (5.7)$$

To calculate the ordinary effective potential $V(\phi; m^2)$, we shift the scalar fields by a constant amount, $\varphi_a = \tilde{\varphi}_a + \delta_{a1}\phi$. To be consistent, the v -vector has to be chosen as $v_a = \delta_{a2}v$. Up to one-loop order, the renormalized ordinary effective potential is

$$\begin{aligned} V(\phi) = & \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^4 + \frac{1}{4(4\pi)^2} \left[(m_1^2)^2 \left(\ln \frac{m_1^2}{\bar{\mu}^2} - \frac{3}{2} \right) + 3(e^2\phi^2)^2 \left(\ln \frac{e^2\phi^2}{\bar{\mu}^2} - \frac{5}{6} \right) \right. \\ & \left. - 2(e^2v\phi)^2 \left(\ln \frac{-e^2v\phi}{\bar{\mu}^2} - \frac{3}{2} \right) + r_1^2 \left(\ln \frac{r_1}{\bar{\mu}^2} - \frac{3}{2} \right) + r_2^2 \left(\ln \frac{r_2}{\bar{\mu}^2} - \frac{3}{2} \right) \right] \end{aligned} \quad (5.8)$$

where $m_1^2 = m^2 + 3\lambda\phi^2$, $m_2^2 = m^2 + \lambda\phi^2$, and $-r_1$ and $-r_2$ are the roots of $(k^2/\xi + e^2\phi^2)(k^2 + m_2^2 + e^2v^2/\xi) - e^2k^2(\phi + v/\xi)^2$. The one- and two-loop order vacuum energy counterterms are

$$\frac{m^4}{4}c_1 = \frac{m^4}{2(4\pi)^2\epsilon} \quad (5.9)$$

and

$$\frac{m^4}{4}c_2 + \frac{m^2v^2}{2}d_2 = \left(\frac{m}{4\pi}\right)^4 \left[\frac{2\lambda}{\epsilon^2} - \frac{3e^2}{2\epsilon^2} + \frac{2e^2}{\epsilon} \right] + \frac{e^4m^2v^2}{(4\pi)^4} \left[\frac{3}{4\epsilon^2} + \frac{1}{4\epsilon} \right]. \quad (5.10)$$

Proceeding as in Sec. IV, we find that to one-loop order the effective potential for the

composite operator is

$$U(\sigma) = \frac{1}{2}m^2\sigma + \frac{\lambda}{4}\sigma^2 + \frac{1}{4(4\pi)^2} \left[4\lambda^2\sigma^2 \left(\ln \frac{2\lambda\sigma}{\bar{\mu}^2} - \frac{3}{2} \right) + 3e^4\sigma^2 \left(\ln \frac{e^2\sigma}{\bar{\mu}^2} - \frac{5}{6} \right) \right], \quad (5.11)$$

while the function f_1 is

$$f_1(\varphi_a\varphi_a) = -\frac{1}{(4\pi)^4\epsilon} (m^2 + \lambda\varphi_a\varphi_a). \quad (5.12)$$

The relationship between J and σ and that between ϕ and σ are

$$\begin{aligned} J(\sigma) &= m^2 + \lambda\sigma + \frac{1}{(4\pi)^2} \left[4\lambda^2\sigma \left(\ln \frac{2\lambda\sigma}{\bar{\mu}^2} - 1 \right) + 3e^4\sigma \left(\ln \frac{e^2\sigma}{\bar{\mu}^2} - \frac{1}{3} \right) \right] \\ &= J_0 + J_1, \end{aligned} \quad (5.13)$$

and

$$\begin{aligned} (\phi^2) &= \sigma - \frac{1}{(4\pi)^2} \left[2\lambda\sigma \left(\ln \frac{2\lambda\sigma}{\bar{\mu}^2} - 1 \right) + \frac{m^2 + \lambda\sigma}{\epsilon} \right. \\ &\quad \left. - (\xi e^2\sigma + 2e^2v\sqrt{\sigma}) \left(\ln \frac{-e^2v\sqrt{\sigma}}{\bar{\mu}^2} - 1 \right) \right] \\ &= (\phi^2)_0 + (\phi^2)_1 \end{aligned} \quad (5.14)$$

The two-loop order correction to the effective potential can be written as

$$U_2(\sigma) = \frac{1}{2}J_1(\phi^2)_1 - \frac{\lambda}{4}(\phi^2)_1^2 + Y_2(\sigma, J_0, (\phi^2)_0), \quad (5.15)$$

where

$$Y_2(\sigma, J, \phi^2) = V_2(\phi^2; m^2 - J) - \frac{J}{2} \left[f_2(\phi^2) + \frac{1}{2}Jc_2 \right] + V_{\text{ex}}(J, \phi^2) \quad (5.16)$$

The term $V_{\text{ex}}(J, \phi^2)$, from insertions of f_1 in one-loop order in one-loop graphs, is

$$\frac{\lambda J}{2(4\pi)^2\epsilon} \left(\int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2 + m_1^2} + \int \frac{d^dk}{(2\pi)^d} \frac{k^2 + \xi e^2\phi^2}{(k^2 + r_1)(k^2 + r_2)} \right), \quad (5.17)$$

while C_2 is given in Eq. (5.10). The total divergent part of $U_2(\sigma)$ is

$$\begin{aligned} & \frac{\lambda^2 \sigma}{\epsilon} \left(-2 \ln \frac{2\lambda\sigma}{\bar{\mu}^2} + \ln \frac{-e^2 v \sqrt{\sigma}}{\bar{\mu}^2} \right) - \frac{3e^4 \sigma}{2\epsilon} \ln \frac{e^2 \sigma}{\bar{\mu}^2} - \frac{9\lambda m^2}{4\epsilon^2} + e^2 m^2 \left(\frac{3}{\epsilon^2} - \frac{2}{\epsilon} \right) \\ & + \lambda^2 \sigma \left(-\frac{13}{4\epsilon^2} + \frac{2}{\epsilon} \right) + \frac{e^4 \sigma}{2\epsilon} + \lambda e^2 \sigma \left(\frac{3}{\epsilon^2} - \frac{2}{\epsilon} \right) + \xi \lambda e^2 \sigma \left(\frac{1}{2\epsilon^2} - \frac{1}{2\epsilon} \right) + \frac{\lambda d^2 v \sqrt{\sigma}}{\epsilon^2} \\ & - \frac{(4\pi)^4}{2} f_2(\sigma). \end{aligned} \quad (5.18)$$

This divergent part is zero in $\overline{\text{MS}}$ scheme. This condition determines the function f_2 uniquely. With this f_2 , the two-loop order correction to the effective potential, $U_2(\sigma)$, is finite:

$$U_2(\sigma) = V_2((\phi^2)_0; m^2 - J_0) + G_1 + G_2 \quad (5.19)$$

where G_1 is the finite part of $\frac{1}{2}J_1(\phi^2)_1 - \frac{\lambda}{4}(\phi^2)_1^2$ and G_2 is finite part of Eq. (5.17).

Let us examine the gauge dependence of $U_2(\sigma)$. To calculate $\frac{\partial V_2}{\partial \xi}$, we will utilize the Nielsen identity[2]

$$\xi \frac{\partial V}{\partial \xi} = C(J, \phi^2, \xi) \frac{\partial V}{\partial \xi}. \quad (5.20)$$

Since the leading order of the function $C(J, \phi^2, \xi)$ is one-loop order, we have

$$\begin{aligned} \xi \frac{\partial V_0}{\partial \xi} &= 0, \\ \xi \frac{\partial V_1}{\partial \xi} &= C_1 \frac{\partial V_0}{\partial \phi}, \\ \xi \frac{\partial V_2}{\partial \xi} &= C_1 \frac{\partial V_1}{\partial \phi} + C_2 \frac{\partial V_0}{\partial \phi}. \end{aligned} \quad (5.21)$$

Using the second equation in Eq. (5.21) and take the limit of $J = J_0$ and $\phi^2 = \phi_0^2$, we get

$$C_1(J_0, \phi_0^2, \xi) = -\frac{\xi e^2 \sqrt{\sigma}}{2(4\pi)^2} \left(\ln \frac{-e^2 v \sqrt{\sigma}}{\bar{\mu}^2} - 1 \right). \quad (5.22)$$

We want to evaluate $\frac{\partial V_2}{\partial \xi}$ at the point $J = J_0$ and $\phi^2 = (\phi^2)_0$. At this point we have

$$\left. \frac{\partial V_0}{\partial \phi} \right|_{J_0, \phi_0^2} = 0. \quad (5.23)$$

Using the result for C_1 , we have

$$\begin{aligned}
\left. \frac{\partial V_2}{\partial \xi} \right|_{J_0, \phi_0^2} &= \frac{1}{\xi} C_1(J_0, \phi_0^2, \xi) \left. \frac{\partial V_1}{\partial \phi} \right|_{J_0, \phi_0^2} \\
&= -\frac{e^2 \sigma}{2(4\pi)^4} \left(\ln \frac{-e^2 v \sqrt{\sigma}}{\bar{\mu}^2} - 1 \right) \left[6\lambda^2 \sigma \left(\ln \frac{2\lambda \sigma}{\bar{\mu}^2} - 1 \right) + 3e^4 \sigma \left(\ln \frac{e^2 \sigma}{\bar{\mu}^2} - \frac{1}{3} \right) \right. \\
&\quad \left. - e^2 \lambda (\xi \sigma + 2v\sqrt{\sigma}) \left(\ln \frac{-e^2 v \sqrt{\sigma}}{\bar{\mu}^2} - 1 \right) \right].
\end{aligned} \tag{5.24}$$

The gauge dependence of G_1 is

$$\begin{aligned}
\frac{\partial G_1}{\partial \xi} &= \frac{1}{2} \left[J_1 - \lambda (\phi^2)_1^{\text{fin}} \right] \frac{\partial (\phi^2)_1^{\text{fin}}}{\partial \xi} \\
&= \frac{e^2 \sigma}{2(4\pi)^4} \left(\ln \frac{-e^2 v \sqrt{\sigma}}{\bar{\mu}^2} - 1 \right) \left[6\lambda^2 \sigma \left(\ln \frac{2\lambda \sigma}{\bar{\mu}^2} - 1 \right) \right. \\
&\quad \left. + 3e^4 \sigma \left(\ln \frac{e^2 \sigma}{\bar{\mu}^2} - \frac{1}{3} \right) - \lambda (\xi e^2 \sigma + 2e^2 v \sqrt{\sigma}) \left(\ln \frac{-e^2 v \sqrt{\sigma}}{\bar{\mu}^2} - 1 \right) \right].
\end{aligned} \tag{5.25}$$

As we can see

$$\left. \frac{\partial V_2}{\partial \xi} \right|_{J_0, \phi_0^2} + \frac{\partial G_1}{\partial \xi} = 0. \tag{5.26}$$

This is a consequence of the Nielsen identity applied to the ordinary effective potential. To see this, let us define

$$\tilde{Y}(\sigma, J, \phi^2) = \frac{1}{2} J \sigma + V(\phi^2; m^2 - J). \tag{5.27}$$

We find that the solution $J_0 + J_1$ and $\phi_0^2 + (\phi^2)_1^{\text{fin}}$ minimizes $\tilde{Y}(\sigma, J, \phi^2)$ to two-loop order. At this point,

$$\frac{\partial \tilde{Y}}{\partial \xi} = \frac{\partial \tilde{Y}}{\partial J} \frac{\partial J}{\partial \xi} + \frac{\partial \tilde{Y}}{\partial \phi^2} \frac{\partial \phi^2}{\partial \xi} + \frac{\partial \tilde{Y}}{\partial \xi} = \frac{\partial V}{\partial \xi}. \tag{5.28}$$

By Nielsen identity, at this point we have

$$\frac{\partial V}{\partial \xi} = \frac{C}{\xi} \frac{\partial V}{\partial \phi} = 0. \tag{5.29}$$

Therefore \tilde{Y} at $J = J_0 + J_1$ and $\phi^2 = \phi_0^2 + (\phi^2)_1^{\text{fin}}$ are gauge independent to two-loop order.

Thus, the two-loop order contribution to \tilde{Y} ,

$$V_2 + G_1 = V_2(\phi_0^2; m^2 - J_0) + \frac{1}{2} J_1(\phi^2)_1^{\text{fin}} - \frac{\lambda}{4} \left[(\phi^2)_1^{\text{fin}} \right]^2, \quad (5.30)$$

is gauge independent as shown in Eq. (5.26). So the gauge dependence of $U(\sigma)$ at two-loop order comes from G_2 , which is the finite part of insertions of f_1 in one-loop graphs. The term $\frac{\partial G_2}{\partial \xi}$ is the finite part of

$$\frac{\lambda(m^2 + \lambda\sigma)}{2(4\pi)^2\epsilon} \int \frac{d^d k}{(2\pi)^d} \frac{e^2\sigma}{(k^2 - e^2 v \sqrt{\sigma})^2}. \quad (5.31)$$

We find that the finite part is

$$\begin{aligned} \frac{\partial G_2}{\partial \xi} = \frac{\lambda e^2 \sigma (m^2 + \lambda\sigma)}{2(4\pi)^4} & \left[\frac{1}{2} \ln^2 \frac{-e^2 v \sqrt{\sigma}}{\mu^2} + \frac{1}{2} \ln^2 4\pi + \beta \right. \\ & \left. - \gamma \ln \frac{-e^2 v \sqrt{\sigma}}{4\pi \mu^2} - \ln 4\pi \ln \frac{-e^2 v \sqrt{\sigma}}{\mu^2} \right]. \end{aligned} \quad (5.32)$$

For the composite effective potential, we have the following result

$$\frac{\partial U_2(\sigma)}{\partial \xi} = \frac{\partial G_2}{\partial \xi} \neq 0. \quad (5.33)$$

When the extra counterterms to the operator $[\varphi_a \varphi_a]_s$ at one-loop order are inserted to one-loop graphs, they cause gauge-dependent contributions to the effective potential $U(\sigma)$ at two-loop level.

However, we can modify our scheme to make the effective potential $U(\sigma)$ gauge independent. We will add and adjust finite terms to the operator $[\varphi_a \varphi_a]_s$ order by order in the perturbation expansion. In the scheme where finite parts vanish, we have shown that the effective potential $U(\sigma)$ is gauge independent at tree and one-loop level. Suppose in zero-finite-part scheme, at all levels of $n - 1$ loops and less, the effective potential is gauge independent, and at n -loop order the effective potential becomes gauge dependent. We can

add a finite counterterm $F_n(\varphi_a\varphi_a)$ to the operator $[\varphi_a\varphi_a]_s$. In this new scheme, the effective potential becomes

$$U(\sigma) + F_n(\sigma) \tag{5.34}$$

at order n . We can choose the function F_n to cancel any gauge-dependent piece of the effective potential $U(\sigma)$ to make the effective potential gauge independent. We can go on to carry out this procedure at higher orders. In this new scheme, the effective potential will be gauge independent. We must stress that since one can always add finite terms to make any gauge-dependent quantities gauge-independent, and there is no preferred prescription for choosing the finite part, this new modified scheme is not very useful in practice.

6. Conclusion

We have demonstrated our method of calculating the composite effective potential in three and four dimensions. It is straightforward to generalize our method to different number of dimensions. In one dimension, the operator φ^2 is finite and we can use it directly in our calculation of composite effective potential. In two and three dimension, this operator becomes divergent and we need to use the renormalized operator by subtracting a divergent quantity from this operator. It is easy to see that the composite effective potential is gauge independent in three or fewer dimensions because of the Nielsen identity. However, in four dimensions, there is no finite effective potential for the conventionally defined composite operator $[\varphi_a\varphi_a]_r$ because graphs with two insertions of this operator remain divergent. Nevertheless, we find that a finite effective potential exists for a modified composite operator in four dimensions. The modified composite operator is the sum of the conventionally defined renormalized operator and some new counterterms. By adjusting the counterterms order by order in a perturbative scheme, we can make the composite effective potential finite. However, the counterterms in the new operator $[\varphi_a\varphi_a]_s$ are no longer purely polynomial in the elementary fields. In a scheme where all counterterms are pure poles in $4 - d$, this finite effective potential is gauge dependent because of the extra counterterms in $[\varphi_a\varphi_a]_s$. We have shown the gauge dependence explicitly at two-loop order.

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REFERENCES

1. L. Dolan and R. Jackiw, Phys. Rev. D**9**, 2904 (1974).
2. N. K. Nielsen, Nucl. Phys. **B101**, 173 (1975).
3. W. Buchmüller, Z. Fodor and A. Hebecker, Phys. Lett. **331B**, 131 (1994).
4. For some previous work, see K. Cahill, Phys. Rev. D**52**, 4704 (1995); K. Okumura, hep-th/9404102; and S. Yokojima, Phys. Rev. D**51**, 2996 (1995).
5. J. M. Cornwall, R. Jackiw and E. Tomboulis, Phys. Rev. D**10**, 2428 (1974).
6. R. Jackiw, Phys. Rev. D**9**, 1686 (1974).
7. J. C. Collins, Nucl. Phys. **B80**, 341 (1974).
8. J. Collins, *Renormalization*, Cambridge University Press, Cambridge, 1984.

Figure 1

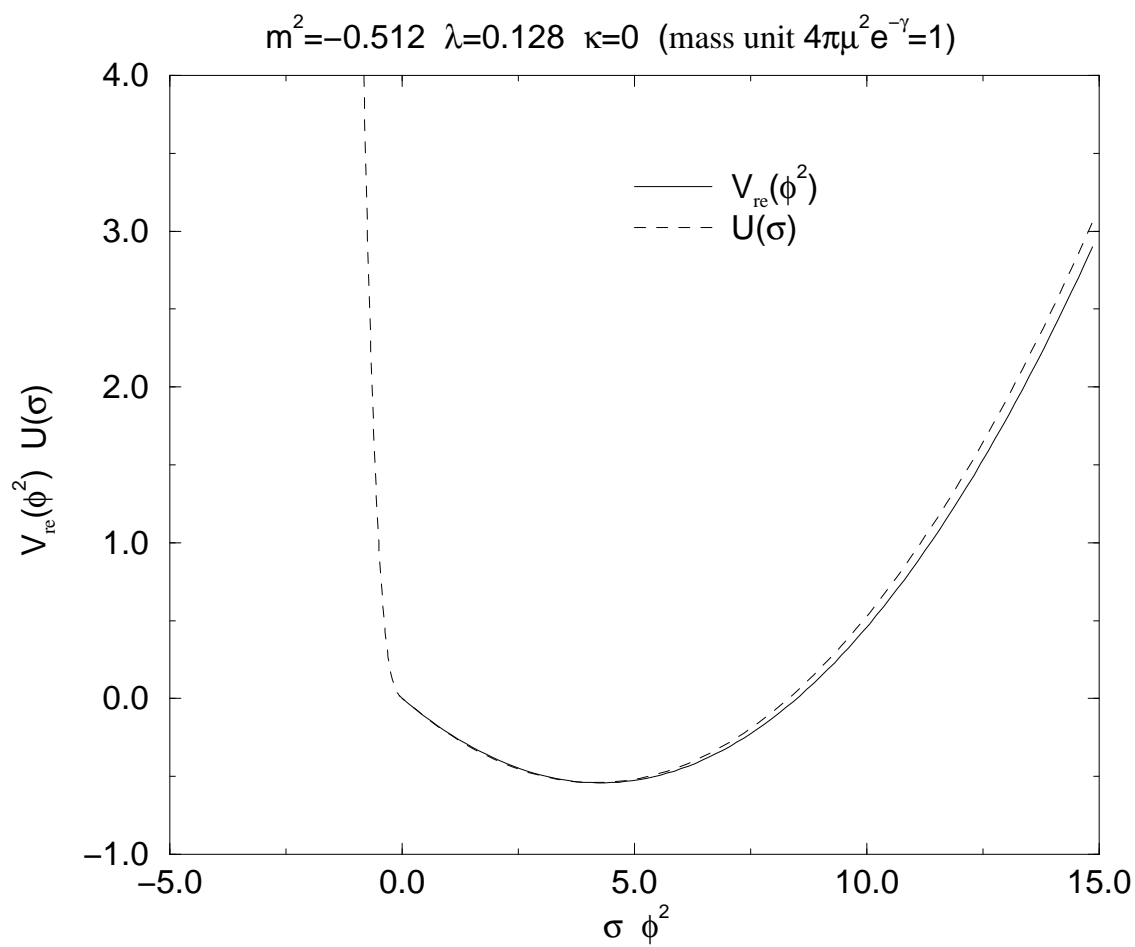


Figure 2

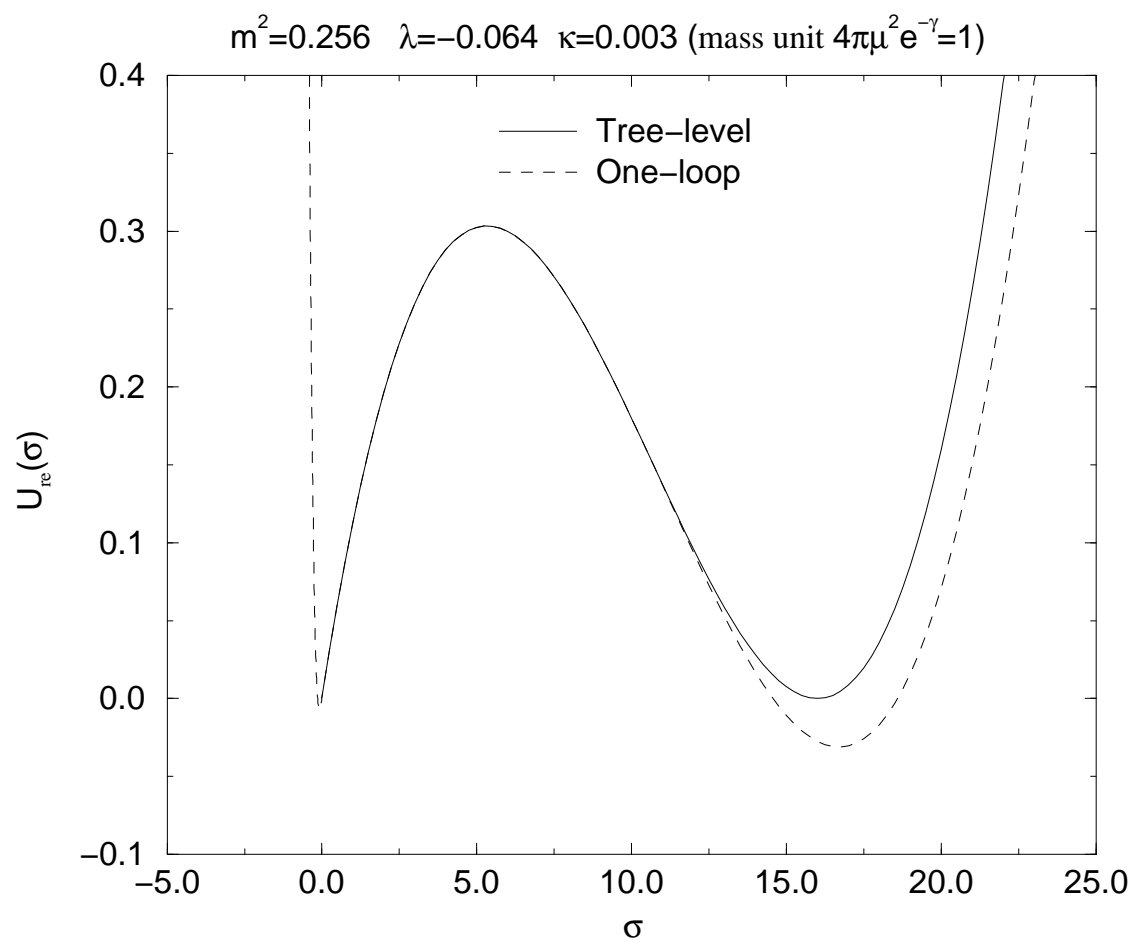


Figure 3

